RESEARCH ARTICLE

ScienceAsia 29 (2003): 279-289

Random Walk on a Plane-Spin-Rotator System: Continuum Theory and Monte Carlo Simulations

Wannapong Triampo,[a,d,*] Darapond Triampo,[b] I-Ming Tang,[a,d] and Yongwimon Lenbury[c]

[a] Department of Physics, Faculty of Science, Mahidol University, Bangkok 10400, Thailand.
[b] Department of Chemistry, Faculty of Science, Mahidol University, Bangkok 10400, Thailand.
[c] Department of Mathematics, Faculty of Science, Mahidol University, Bangkok 10400, Thailand.
[d] Capability Building Unit in Nanoscience and Nanotechnology, Faculty of Science, Mahidol University, Bangkok 10400, Thailand.
* Corresponding author, E-mail: wtriampo@yahoo.com

Received 2 Jan 2003
Accepted 21 Feb 2003

ABSTRACT: We introduce a Plane-Spin-Rotator (PSR) model as one of a myriad of non-equilibrium statistical mechanics models governed by stochastic dynamics. The system consists of a one-dimensional chain of lattice sites in which each site is attached with a spin initially in the ordered state e.g., pointing in the same direction. We incorporate the effects of a non-equilibrium phenomenon by giving the system a dynamics. Namely we put a random walker (RW) or a Brownian agent at the origin or in the middle of the lattice chain at the beginning and let it execute pure unbiased random walk to disorder i.e., to destroy the line up of the spins. The local update rule whereby the system changes periodically from one state to another is that each time step as the RW moves it has a certain probability to rotate the spin or change the angle $\Theta(x,t)$ between the x-axis and the spin. We find the nontrivial statistics $\Theta(x,t)$ of due to this and other simple stochastic (Markovian type) model such as $\langle \cos\Theta(0,t)\rangle$ and $\langle \sin\Theta(0,t)\rangle$ do not behave in sinusoidal fashion as one might expect. These functions and other can be calculated analytically exploiting the results from its "cousin" model introduced in *Physical Review E, Vol. 59, no. 5 p.5127*. Excellent agreement from theoretical and Monte Carlo computer simulation results is found.

KEYWORDS: Stochastic process, Random walk, Brownian, spin, Monte Carlo, non-equilibrium statistical mechanics

INTRODUCTION

Random walk or Brownian motion (continuous limit counter part) is one of the fundamental processes in nature. Originally observed in the jiggly irregular motion of pollen grains suspended in water by the English botanist Robert Brown, it was first cast into the mathematical language by Einstein in 1905.[1] Random walk is one of the most studied problems and versatile concepts in statistical physics. Reif[2] used random walk to introduce many of the basic and essential concepts of statistical mechanics, while Feller[3] utilized the same subjects to illustrate the concepts of probability. To physicists and mathematicians, random walk is a basic paradigm in stochastic processes and is worthy of many monographs. The most recent ones are those by Weiss[4] and Hughes.[5] The theory of random walk has attracted much theoretical attention over the past sixty years due to its numerous applications in the physics,[6] astronomy,[7] chemistry,[8] biological science,[9] and even social science.[10] The reason for the multiple connections between random walk and many different questions of current and permanent interest in science is the mathematics. A very rich field of research has built up around the behavior of a random walker (RW) interacting with an environment, a good example being the diffusion of electrons in disordered medium.[11] One can also consider the RW to be the disordering agent in its environment.[12-13]

Thanks to the detailed studies of the data corruption (DC) we found an interesting application and have casted into the new model presented in this paper. We shall call it the Plane-Spin-Rotator model (PSR model). Briefly, this model consists of an RW in one dimension lattice. Each lattice site element is described by the spin lying in a plane like a clock or a rotator, initially all pointing in the same direction. As the RW wanders through, it has a certain probability to rotate the spin or change the angle between the x-axis and the spin direction, $\Theta(x,t)$. The RW is not affected by the environment in any way. Thus if we start with a system in which all spins exist in the same state (e.g. zero $\Theta(x,t)$) or the *ordered* initial configuration and introduce the RW at origin x = 0, then after some time, there will be a region around the origin in which the elements will be found in the mixture of various values of . An interesting question concerns the degree of disordering which exists for elements within this region namely x-component cos( ) and y-component sin($\Theta(x,t)$) (which are projections of the spin on to the x and y axes,respectively). We find that the statistics due to this such simple stochastic model is nontrivial and can be derived analytically by exploiting the results from DC.

The paper is organized as follows. In Section II, we recap our previous work on DC. In Section III, we define our model and discuss which quantities of interest is to be investigated. We then turn to our findings. In Section IV we calculate the predicted analytic results obtained in the first subsection and compare it with the detailed Monte Carlo in the following subsection. Finally, we summarize and present some comments and open questions.

Recapitulation

In this section we give a very brief review of the main ideas and some of the results contained in DC that will be of use in the present work. The process of random walk in a binary medium is first modeled on a hypercubic lattice of dimension *d*. A position of the RW is denoted by a lattice vector $\vec{R}(t)$. In a time step , the RW has a probability *p* to move to one of its *2d* nearest neighbor sites. In making such a jump, there is a probability *p* that the element on the site will switch to a new value. The elements are described by spin variables $\sigma_{\vec{r}}$ (where $\vec{r}$ denotes a discrete lattice vector) which may take the values (see Fig. 1). The spin variables encode the information about the disordering process. For example in the data corruption process we label uncorrupted bits (of value 1) by spin +1 and corrupted bits (of value 0) by -1. (We will often use the terms "magnetization density" and "global magnetization", which may be simply translated to "density of disorder" and "total amount of disorder", respectively.)

We can define the dynamics via the probability distribution which is the probability that at time t, the RW is at position and the spins have values given by the set . This distribution evolves according to a master equation[14] which takes the form

$$P(\vec{R},\{\sigma_{\vec{r}}\},t+\delta t) = \qquad + \frac{p(1-q)}{2d}\sum_{\vec{l}} P(\vec{R}+\vec{l},\{\sigma_{\vec{r}}\},t) + \frac{pq}{2d}\sum_{\vec{l}} P(\vec{R}+\vec{l},\ldots,-\sigma_{\vec{R}+\vec{l}},\ldots,t) \qquad 1$$

where $\{\vec{l}\}$ represents the orthogonal lattice vectors (which have magnitude *l*).

In DC an alternative continuum description was obtained by viewing the process as a stochastic cellular automaton (SCA).[15] The process is then defined in terms of the position $\vec{R}(t)$ of the RW, and the coarse-grained density of disorder (or magnetization density), which is defined in a small region of space at a specific time, is a functional of . In some sense, one may view this in the same spirit as a Langevin description of a stochastic process described at a more fundamental level by a master equation. Taking the continuum limit of this description yields a simple Langevin equation for the position of the RW:

$$\qquad 2$$

where $\vec{\xi}(t)$ is a noise term, each component of which is an uncorrelated Gaussian random variable with zero mean (i.e. is a white noise process). The correlator of $\vec{\xi}(t)$ is given by

$$= D\delta_{i,j}\delta(t-t') \qquad 3$$

Here and from now on, angled brackets indicate an average over the noise (or equivalently the paths of the RW). The RW is chosen to reside initially at the origin: The evolution of the magnetization density ϕ is described by

$$= - \qquad 4$$

This equation may be integrated to give the explicit functional solution

$$\varphi(\vec{r},t) = \exp[-\lambda\int_0^t dt'\Delta_{\vec{l}}(\vec{r}-\vec{R}(t')) \qquad 5$$

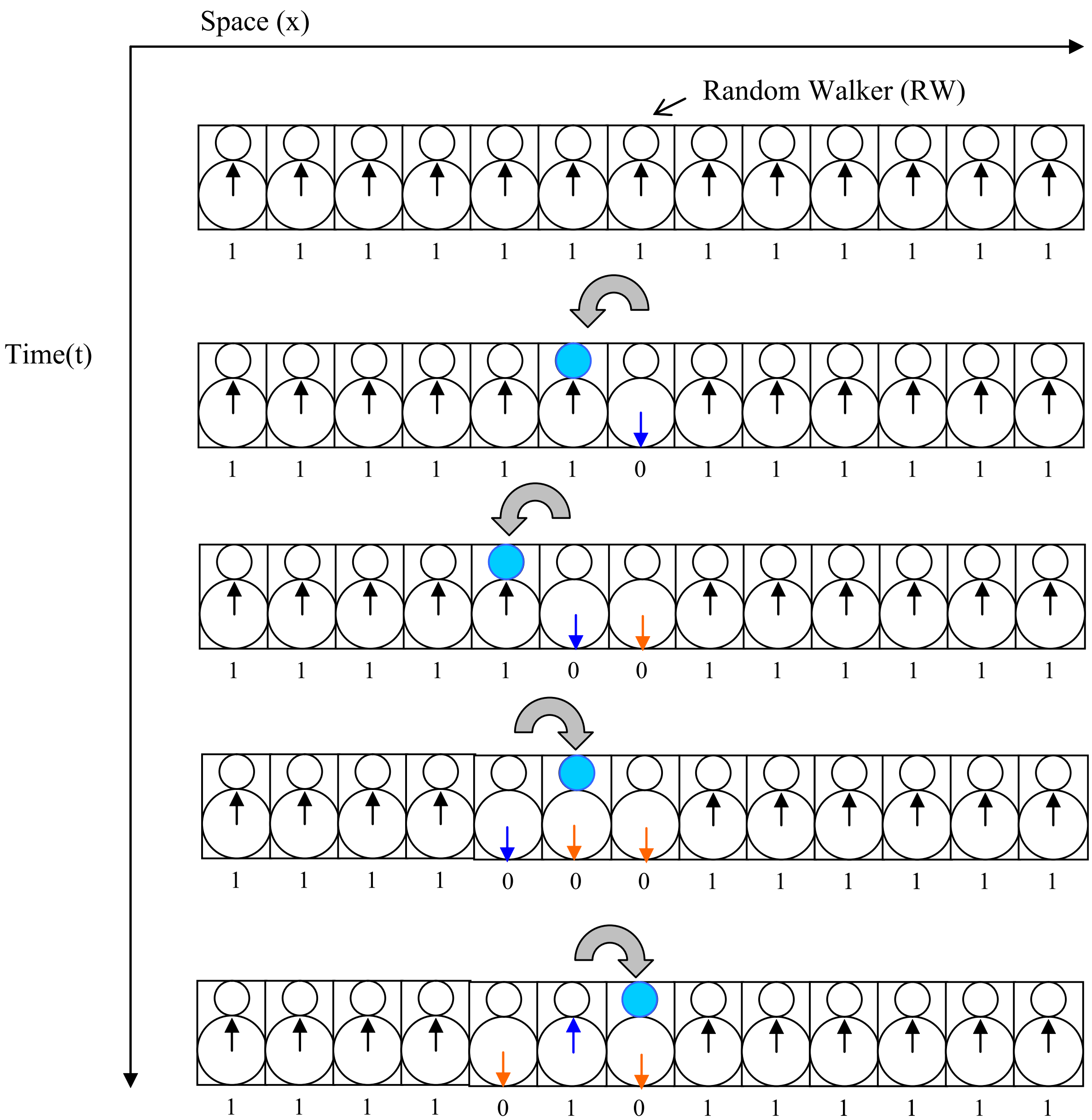
The above solution is obtained for an initial condition $\varphi(\vec{r},0) = 1$ which we shall use exclusively. In terms of the original lattice model, it corresponds to choosing all the spins to have the initial value of +1, so that we measure the subsequent disorder of the system by counting the number of minus spins in the system.

In DC we only use the continuum description to generate results for various average quantities. The simplest quantity to consider is the mean magnetization density for d=1 given by

$$m(\vec{r},t) = \langle\varphi(\vec{r},t)\rangle = \sum_{n=0}^{\infty}(-\lambda)^n\chi_n(x,t) , \qquad 6$$

where $\chi_0(x,t) = 1$, and for $n > 0$,

$$. \qquad 7$$



$d = 1$

Fig 1. Illustration of the data corruption processes for and $q = 1$. The initial uncorrupted state is shown on the top, with the RW represented by the filled circle. From the top, we show a typical walk of 4 steps. The RW switches a bit each visit, so those bits visited an even number of times are restored to their original value.

In DC, it can be shown that

$$\chi_n(x,t) = \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \ldots \int_0^{\tau_{n-1}} d\tau_n g(0,\tau_1-\tau_2)\ldots \times g(0,\tau_{n-1}-\tau_n) g(x,\tau_n) \qquad 8$$

where $g(x,t) = (2\pi Dt)^{-1/2}\exp(-x^2/2Dt)$ is the probability density of random walk. Due to the structure of Eq (8) which is an n-fold convolution, we apply the temporal Laplace transform and obtain for n>0

$$\tilde{\chi}_n(x,s) \equiv \int_0^\infty dt e^{-st}\chi_n(x,t) = \frac{1}{s}\tilde{g}(0,s)^{n-1}\tilde{g}(x,s), \qquad 9$$

where

$$\tilde{g}(x,s) = \frac{1}{(2DS)^{1/2}}\exp\left[-\left(\frac{2S}{D}\right)^{1/2}|x|\right]. \qquad 10$$

where $\tilde{g}(x,s)$ is the Laplace transform of diffusion equation Green function. Summing over these function as formulated in Eq. (6) we find

. 11

This exact result allows one to extract a great deal of statistical information about the process. First, one can simply invert the Laplace transform to find the average magnetization density (or average density of disorder relative to 1/2) as a function of $\vec{r}$ and *t* Explicit forms are given in DC. For , the form is

$$m(x,t) = \mathrm{erf}\left[\frac{|x|}{(2Dt)^{1/2}}\right] + \exp\left(\frac{\lambda|x|}{D} + \frac{\lambda^2 t}{2D}\right) \times \mathrm{erfc}\left[\lambda\left(\frac{t}{2D}\right)^{1/2} + \frac{|x|}{(2Dt)^{1/2}}\right] \qquad 12$$

where and $\mathrm{erfc}(z)$ are error function.[15] Considering the long time behavior of the above expression, we find that the average magnetization density at the origin $(x=0)$ decays asymptotically as

$$= \left(\frac{2D}{\pi\lambda^2 t}\right)^{\frac{1}{2}}\left[1 + O\left(\frac{D}{\lambda^2 t}\right)\right] \qquad 13$$

We note here that the continuum solution has the important property that $\langle \varphi(\vec{r},t;\lambda)^n \rangle$ = . This allows us to utilize the exact solution to reconstruct the probability density for the magnetization density.

Model

In this section we shall formulate the PSR model using the continuum theory as in DC. It is obvious as we mention in DC that formulation using the discrete theory and continuum theory is equivalent. We also want to stress that this is just the alternative method of characterizing the evolution of the system other than using the evolution of the probability distribution of the configuration via the master equation. We start by defining = ≡ as spin variable. Since we are interested in how the angle $\Theta(x,t)$ changes with time as the RW wandering through the lattice system, we use the dynamics similar to that use in DC except now the RW interacts with the lattice by rotating the spin as the result of changing $\Theta(x,t)$ (for $d=1$, see Fig. 2). The local update rule which causes the system to change periodically from one state to another is that each time step as the RW makes a random jump to one of its (*2d*) nearest neighbors, the RW has a certain probability (here p=1, q=1) to rotate the spin or change the angle between the x-axis and the spin, $\delta\Theta(x,t)$. We write the local rules for the process in the spirit of the SCA. The local rules for such process are easily written down as

$$R(t+\delta t) = R(t) + l(t) \qquad 14$$

$$\Theta(x,t+\delta t) = \Theta(x,t) + \delta_{x,R(t)} \qquad 15$$

We are interested in a continuum limit of these two rules. The first is nothing more than a random walk. We take the lattice position R(t) to be a continuum quantity, and we replace the random unit lattice position l(t) by continuum , which is an uncorrelated Gaussian random variable (a white noise process) reflecting the random process that has zero configuration average or average over the noise (or equivalent to the paths of RW).

$$\langle \xi(x,t) \rangle = 0 \qquad 16$$

The correlator or the second moment of the noise is given by

$$\langle \xi(x,t)\xi(x',t') \rangle = D\delta(x-x')\delta(t-t') \qquad 17$$

Relation (11) implies that the noise has no correlations in space and time. Then, on taking $\delta t \to 0$, equation (10) assumes the form

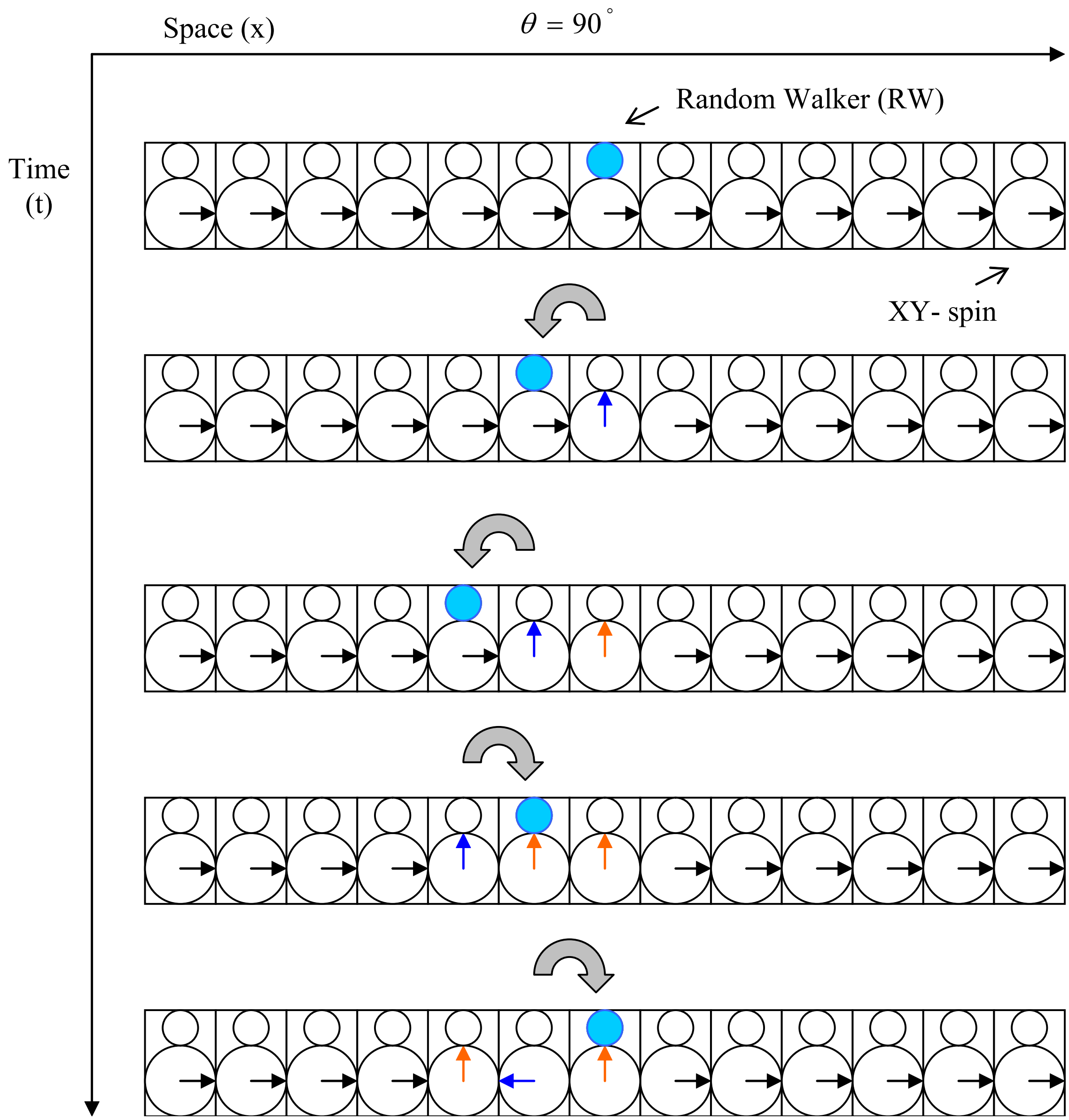


Fig 2. Illustration of the Plane-Spin-Rotator model for $d = 1$. The initial $(t = 0)$ ordered state is shown on the top with the RW represented by the filled circle and the spin is underneath of each RW. On downward order we show a typical (1 from 2^5) 5 steps.

$$\frac{dR}{dt} = \xi(t) \qquad 18$$

which is the familiar equation for a continuum random walk where D is the diffusion constant.[14] The second SCA rule is more complicated to generalize to continuum. Moving the first term over to the left-hand side which may then be taken to be time derivative of . The rest piece resembles a constant term centered at x = R

$$\partial_t \Theta(x,t) = \qquad 19$$

where λ is a phenomenological parameter which describes how strongly the spin is coupled to the RW. We stress that the field Θ is a function of the continuous space and time variables x and *t*,respectively, and a function of the path R(t) of the RW. As we pointed out in DC paper that this continuum equation (15) is not strictly well derived from Eq. (11), as we have not rigorously proved that the continuum limit exists. In fact, we shall find that for , the lattice scale is crucial, and consequently we must soften the Dirac function to a better defined sharply peaked function.

However, from the following results we will see good agreement between the data and the analytic results to support our assumption.

Having shown a heuristic derivation of the continuum theory based on the SCA for the case in which the RW always moves, we proceed to the next section in which we present a comprehensive solution in one dimension.

Results

In this section we will present some analytic results and some detailed Monte Carlo simulation results.

Analytic results

Here, we are interested in solving Eq. (15) and analyze how the angle $\Theta(x,t)$ evolves with time. Thanks to the nature of the model, one of the positive features of the continuum theory described by the Eq. (15) is that one may immediately integrate the equation to find as an explicit function of the path of the RW. We first perform a dimensional analysis to be used as a reference.

Letting L be the dimension of length and T be the dimension of time, the dimensions of the significant terms are

$$[R] = L$$

$$[\xi^2] = \frac{L^2}{T^2} = \frac{[D]}{T} \Rightarrow [D] = \frac{L^2}{T}$$

$$[\partial_t \Theta] = \frac{1}{T} = \frac{[\lambda]}{L} \Rightarrow [\lambda] = \frac{L}{T}$$

$$\left[\frac{\lambda^2 t}{D}\right] = \frac{L^2}{T^2} T \frac{T}{L^2} = 1$$

$$\left[\frac{|x|\lambda}{D}\right] = L.\frac{L}{T}\frac{T}{L^2} = 1$$

$$\left[\frac{|x|}{\lambda t}\right] = \frac{L}{T}\frac{T}{L} = 1$$

Next, we integrate Eq. (15), and get

$$\Theta(x,t) = \lambda \int_0^t dt' \delta(x - R(t')) . \qquad 20$$

This solution requires the initial condition $\Theta(x,0) = 0$. It is important to note here that $\Theta(x,t)$ is non-negative for all *x* and *t*. To get the stochastic properties that we want, we connect these results with those in DC given in section 2. To do this, we set up the solution to Eq. (16) as

$$\Omega(x,t) \equiv \left\langle e^{i\Theta(x,t)} \right\rangle = \left\langle \exp\left[i\lambda \int_0^t dt' \delta(x - R(t')) \right] \right\rangle . \qquad 21$$

We rewrite Eq. (5) as

$$\left\langle \exp\left[-\tilde{\lambda} \int_0^t dt' \delta(x - R(t')) \right] \right\rangle = \operatorname{erf}\left[\frac{|x|}{(2Dt)^{\frac{1}{2}}} \right] + \exp\left[\frac{\tilde{\lambda}|x|}{D} + \frac{\tilde{\lambda}^2 t}{2D} \right] \operatorname{erfc}\left[\tilde{\lambda}\left(\frac{t}{2D}\right)^{1/2} + \frac{|x|}{(2Dt)^{1/2}} \right], \qquad 22$$

where $\tilde{\lambda}$ is just (we use the new notation to avoid confusion). Comparing Eq. (17) with Eq.(18) and keeping in mind that for this case in d=1 dimension $\Delta(\vec{r} - \vec{R})$ can be replaced by delta function $\delta(x - R)$. The parameter $\tilde{\lambda}$ is now . The now takes the form

$$\Omega(x,t) = \operatorname{erf}\left[\frac{|x|}{(2Dt)^{\frac{1}{2}}} \right] + \exp\left[\frac{-i\lambda|x|}{D} - \frac{\lambda^2 t}{2D} \right] \operatorname{erfc}\left[-i\lambda\left(\frac{t}{2D}\right)^{1/2} + \frac{|x|}{(2Dt)^{1/2}} \right] . \qquad 23$$

Recalling the definition of $\Omega(x,t)$, we note that

$$\operatorname{Re}[\Omega(x,t)] = \langle \cos(\Theta(x,t)) \rangle \equiv \qquad 24\text{-a,}$$

$$\operatorname{Im}[\Omega(x,t)] = \langle \sin(\Theta(x,t)) \rangle \equiv \qquad 24\text{-b,}$$

where $\mathrm{Re}[\Omega(x,t)]$ and $\mathrm{Im}[\Omega(x,t)]$ are the real and imaginary part of $\Omega(x,t)$ respectively. Applying the trigonometric theorem, we get

$$\Omega(x,t) = \mathrm{erf}\left[\frac{|x|}{(2Dt)^{\frac{1}{2}}}\right] + \exp\left[-\frac{\lambda^2 t}{2D}\right]\left\{\cos\left(\frac{\lambda|x|}{D}\right) - i\sin\left(\frac{\lambda|x|}{D}\right)\right\}\cdot$$

$$\left\{\mathrm{erfc}\left(\frac{|x|}{(2Dt)^{1/2}}\right) - \frac{2}{\sqrt{\pi}}e^{-\frac{x^2}{2Dt}}\int_0^{\lambda\left(\frac{t}{2D}\right)^{1/2}} d\beta\, e^{\beta^2}\sin\left(2\beta\frac{|x|}{(2Dt)^{1/2}}\right) + i\frac{2}{\sqrt{\pi}}e^{-\frac{x^2}{2Dt}}\int_0^{\lambda\left(\frac{t}{2D}\right)^{1/2}} d\beta\, e^{\beta^2}\cos\left(2\beta\frac{|x|}{(2Dt)^{1/2}}\right)\right\} \quad \mathbf{25}$$

It is a straightforward matter after some rearrangement

$$\langle\sigma_1(x,t)\rangle = \mathrm{erf}\left[\frac{|x|}{(2Dt)^{\frac{1}{2}}}\right] + \exp\left[-\frac{\lambda^2 t}{2D}\right]\bullet\{\cos\left(\frac{\lambda|x|}{D}\right)\mathrm{erfc}\left(\frac{|x|}{(2Dt)^{1/2}}\right)$$

$$-\cos\left(\frac{\lambda|x|}{D}\right)\frac{2}{\sqrt{\pi}}e^{-\frac{x^2}{2Dt}}\int_0^{\lambda\left(\frac{t}{2D}\right)^{1/2}} d\beta\, e^{\beta^2}\sin\left(2\beta\frac{|x|}{(2Dt)^{1/2}}\right)$$

$$+\sin\left(\frac{\lambda|x|}{D}\right)\frac{2}{\sqrt{\pi}}e^{-\frac{x^2}{2Dt}}\int_0^{\lambda\left(\frac{t}{2D}\right)^{1/2}} d\beta\, e^{\beta^2}\cos\left(2\beta\frac{|x|}{(2Dt)^{1/2}}\right)\} \quad \mathbf{26}$$

After more simplification, we get

$$\langle\sigma_1(0,t)\rangle = e^{-\frac{\lambda^2 t}{2D}}$$

$$\langle\sigma_1(x,t)\rangle = e^{-\frac{\lambda^2 t}{2D}}\cos\left(\frac{\lambda|x|}{D}\right) + \mathrm{erf}\left(\frac{|x|}{(2Dt)^{1/2}}\right)\cdot\left[1 - e^{-\frac{\lambda^2 t}{2D}}\cos\left(\frac{\lambda|x|}{D}\right)\right]$$

$$+\frac{2}{\sqrt{\pi}}\exp\left(-\frac{\lambda^2 t}{2D} - \frac{x^2}{2Dt}\right)\cdot\int_0^{\lambda\left(\frac{t}{2D}\right)^{1/2}} d\beta\, e^{\beta^2}\sin\left(\frac{\lambda|x|}{D} - \frac{2\beta|x|}{(2Dt)^{1/2}}\right) \quad \mathbf{27a}$$

and

$$\langle\sigma_2(x,t)\rangle = -\mathrm{erfc}\left(\frac{|x|}{(2Dt)^{1/2}}\right)e^{-\frac{\lambda^2 t}{2D}}\sin\left(\frac{\lambda|x|}{D}\right)$$

$$+\frac{2}{\sqrt{\pi}}\exp\left(-\frac{\lambda^2 t}{2D} - \frac{x^2}{2Dt}\right)\cdot\int_0^{\lambda\left(\frac{t}{2D}\right)^{1/2}} d\beta\, e^{\beta^2}\cos\left(\frac{\lambda|x|}{D} - \frac{2\beta|x|}{(2Dt)^{1/2}}\right) \quad \mathbf{27b}$$

Considering the specific value of the above expression, we have, for $x = 0$,

, **28a** and $$\langle\sigma_2(0,t)\rangle = \frac{2}{\sqrt{\pi}}\cdot e^{-\frac{\lambda^2 t}{2D}}\cdot\int_0^{\lambda\left(\frac{t}{2D}\right)^{1/2}} d\beta\, e^{\beta^2} \quad \mathbf{28b}$$

Since we wish to compare our analytic results with the Monte Carlo computer simulation results, we consider the two limits;

$$\langle \sigma_2(0,t) \rangle = \begin{cases} \dfrac{2}{\sqrt{\pi}} \cdot \left(\frac{\lambda^2 t}{2D}\right)^{1/2} \cdot \left[1 - \dfrac{2}{3}\frac{\lambda^2 t}{2D} + O\left(\left(\frac{\lambda^2 t}{2D}\right)^2\right)\right]; & \frac{\lambda^2 t}{2D} << 1 \\ \dfrac{1}{\sqrt{\pi}} \cdot \left(\frac{2D}{\lambda^2 t}\right)^{1/2} \left[1 + \dfrac{1}{4}\frac{2D}{\lambda^2 t} + O\left(\left(\frac{2D}{\lambda^2 t}\right)^2\right)\right]; & \frac{\lambda^2 t}{2D} >> 1 \end{cases} \qquad 29$$

From Eq. (25), it can be show that $\langle \sigma_2(0,t) \rangle$ behaves differently for small and large time with a cross over which can be calculated as follows;
let, $\frac{\lambda^2 t}{2D} = a^2$ we rewrite

$$\langle \sigma_2(0,t) \rangle = \frac{2}{\sqrt{\pi}} \cdot e^{-a^2} \cdot \int_0^a d\beta\, e^{\beta^2}, \qquad 30$$

The maximum value of the function can be obtained by differentiating with respect to a, and set it to zero, i.e.,

$$\partial_a \langle \sigma_2(0,t) \rangle \big|_{a^*} = 0 = 1 - 2a^* e^{-a^{*2}} \int_0^{a^*} d\beta\, e^{\beta^2} \qquad 31$$

We get

$\langle \sigma_2(0,t) \rangle_{max} = \dfrac{1}{a^* \sqrt{\pi}}$ which is independent of λ and D. Using some trigonometric function identity, we can show that

$$\begin{aligned} \langle \sigma_1^2(x,t) \rangle &= \langle \cos^2 \Theta(x,t) \rangle = \frac{1}{2}\left\{1 + \langle \cos\ (2\Theta 2x,t)) \rangle\right\} \\ &= \frac{1}{2} + \frac{1}{2}\langle \sigma_1(x,t;2\lambda \rangle \end{aligned} \qquad 32a$$

and

$$\langle \sigma_2^2(0,t) \rangle = \frac{1}{2}\left(1 - e^{-\frac{2\lambda^2 t}{D}}\right) \qquad 32b$$

Again we focus on the value of the function at the origin, we have

$$\langle \sigma_1^2(0,t) \rangle = \frac{1}{2}\left(1 + e^{-\frac{2\lambda^2 t}{D}}\right) \qquad 33a$$

$$\langle \sigma_2^2(0,t) \rangle = \frac{1}{2}\left(1 - e^{-\frac{2\lambda^2 t}{D}}\right) \qquad 33b$$

We now turn to simulation results.

Monte Carlo Simulation Results

Our aim in this section is to show the validity of our theoretical results from the previous section. To do so, we have performed Monte Carlo simulations of the discrete version model, defined in section 2. In all of the simulations for which we present results, we have set the hopping rate *p* of the RW to unity. All of our results are obtained for a one-dimensional chain of sites. The chain length is unimportant, so long as one ensures that the RW never touches the edges in any of its realizations up to the latest time at which data is extracted. We perform an average over 10^6 realizations (or runs). Such simulations require a few days on a DEC Alpha 233 MHz

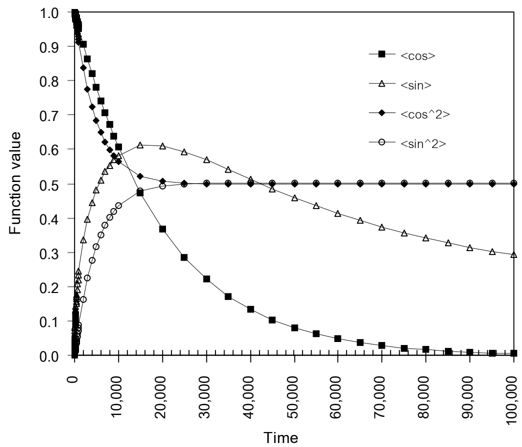
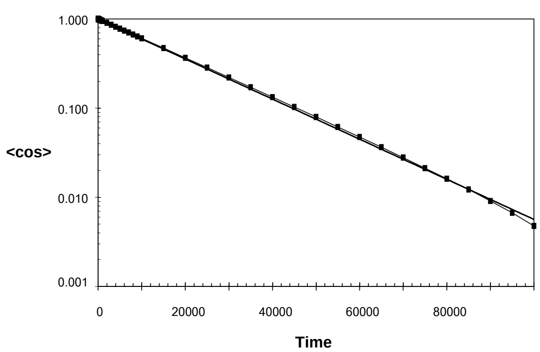


Fig 3. Plot of simulation data for the $\langle \cos\Theta(0,t)\rangle$, $\langle \sin\Theta(0,t)\rangle$, $\langle \cos^2\Theta(0,t)\rangle$, and $\langle \sin^2\Theta(0,t)\rangle$.



$\langle \cos\Theta(0,t)\rangle$

Fig 4. Semilog plot of vs. time. The denser solid line (which is partially obscured by the data) is the regression line with function $1.005e^{-5\times10^{-5}t}$ and $r^2 = 0.9996$

workstation. In a given run, the RW is moved left or right with an equal probability at each time step and the spin it left behind is rotated. Each run starts with the same initial configuration; namely all spins are pointed to the x-direction making zero degree with x-axis.

In Fig 3 we show the numerical results of all the means of trigonometric function of $\Theta(x,t)$ at origin namely , $\langle \sin\Theta(0,t)\rangle$, $\langle \cos^2\Theta(0,t)\rangle$, and $\langle \sin^2\Theta(0,t)\rangle$ versus time. They all appear to have the correct features as theoretically predicted. This confirms the relations given in Eq. (28), Eq. (29), and Eq. (33).

In Fig 4, we show the semi-log plot of $\langle \cos\Theta(0,t)\rangle$ vs. time. Due to the exponentially function feature as analytically predicted it results in the straight line. Using Excel program to do multiple regression, it shows the good fit with the fitting function $1.005e^{-5\times10^{-5}t}$ (r^2 = 0.9996). This emphasizes the agreement between the numerical data and Eq. (28)

In Fig 5, to verify the Eq. (29), we log-log plot of the $\langle \sin\Theta(0,t)\rangle$ function versus time. The plot does show two regimes with different power laws as predicted. In the early regime the graph gives the straight line with slope about 0.5 and the fitted function $0.0082t^{0.4922}$, with r^2 = 0.9982, which is in good agreement with predicted theory. With the cross-over time about the time step 10,000 to 30,000, it turns into another scaling law with another slope (predicted slope = -0.5) as seen in later regime. Due to the problem about the computing resources we now have, we consequently did not perform simulation for the time long enough to get the convincing data (at least one and haft decade of power law or straight line). Therefore from the data they only confirm the two power law behaviors but not confirm the power law of the later regime. Arguably, we did try to extrapolate the curve to see roughly if its slope were -0.5. We found that the slope could be -0.5. However, in the near future we hope to get more powerful computers to run for longer time with a lot of runs to get the nice curve with the least noise.

It should also be mentioned about the lines $\langle \cos^2\Theta(0,t)\rangle$ and $\langle \sin^2\Theta(0,t)\rangle$. Early, they are controlled by the exponential feature and asymptotically approach to 0.5 afterward as predicted by continuum theory. We end this section by mentioning that the agreement between the numerical data and the continuum theoretical results provides a very strong evidence for the validity of our whole continuum approach.

Summary and Conclusion

We have introduced and analyzed the PSR model undergoing disordering, focusing on its stochastic dynamics due to an RW (or a Brownian agent). In section 2, we presented a review of our previous work on DC and described what motivated us to study this model. In section 3, we formulated the continuum equations of our model by starting from the discrete version in the spirit of stochastic cellular automata, which consists of the RW rotating clocks (or spins) on a lattice as it wanders. The model is non-trivial since the value of Θ depends sensitively on the path of the RW *i.e.* how often the RW has visited the spins. The continuum equation formulation of the PSR model has similar feature as in the system which we used before to describe the data corruption model in DC. In section 4, we examined the properties of the continuum theory for $d = 1$. In the first subsection, we derived an exact expression for the evolution of the x and y component of the spin i.e., and $\sin(\Theta(x,t))$. Using a fortuitous property of the original

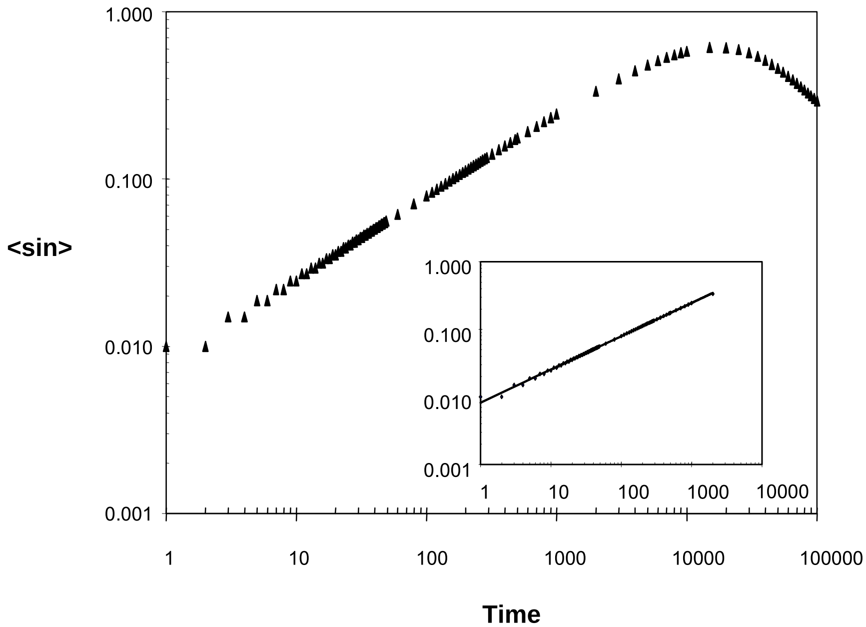


Fig 5. Log-log plot of $\langle \sin \Theta(0,t) \rangle$ vs. time to verify the power law behavior of the data. The inset shows the regression line of curve fitting with function $0.0082t^{0.4922}$ and $r^2 = 0.9982$

continuum theory in DC, we found that at the origin the average of time variation of x-component for small and large time have the following behavior. For the x-component, it decreases exponentially to zero with the characteristic time scale $2D/\lambda^2$. In contrast, the y-component is proportional to $\sqrt{t}$ for small times and goes like $1/\sqrt{t}$ for large time with the maximum independent of the diffusive and the coupling constant. We have calculated the next moment of both quantities namely $\langle \cos^2 \Theta(0,t) \rangle$ and $\langle \sin^2 \Theta(0,t) \rangle$. In the second subsection, we presented our computer experimental results from Monte Carlo simulations of the discrete lattice model. We have measured the temporal variation of all corresponding analytic results predicted. In all cases we found good agreement between our data and the theoretical predictions arising from the continuum model.

In conclusion, we have introduced and solved a model in which the RW interacts with a spin environment. The benefit from this work is of course the insight into one specific example of the stochastic process mediated by a random or Brownian agent. One can view this problem as the disordering process starting from the initially ordered configuration and the degree of disordering increase as time goes on due to the dynamics or the local update rule applied. In DC our primary application is to an environment composed of bits of (two states) data, which the RW steadily corrupts. We can apply our results to a system characterized by more than two states. It remains to be seen whether one can find a solid application of the models. In the future work, we will explore cases of two or more dimensions of agents or walker. There are many future directions for future work, foremost among which are (i) calculating all quantities in two dimensional space (ii) investigating two point an autocorrelation functions in both one and two dimensions (iii) studying many agents and more generalized coupling to make a stronger connection to real processes.

Acknowledgements

We thank N. Nuttawut, A. Kitichattraporn and S. Sa-nguansin for helpful reading of a manuscript and for their comments. We also greatly appreciate both referees for their very helpful comments. This research is supported in part by Thai Research Fund through the grant number TRG4580090 and RTA4580005 and MTEC Young Research Group funding MT-NS-45-POL-14-06-G.

References

1. Einstein A (1956) *Investigations on the Theory of the Brownian Motion*. Dover, USA.
2. Reif F (1965) *Fundamentals of Statistical and Thermal Physics*. McGraw-Hill, New York.
3. Feller W (1968) *An Introduction to Probability Theory and Its Applications*, 3rd ed. Wiley, New York.
4. Weiss GH (1994) *Aspects and Applications of the Random Walk*.

North Holland, Amsterdam.
5. Hughes BD (1995) *Random Walks and Random Environments*. Clarendon Press, Oxford.
6. Chandrasekhar S (1943) Stochastic problems in physics and astronomy. *Rev Mod Phys* **15**, 1.
7. Weiss GH and Rubin RJ (1983) Random walk: theory and selected applications. *Adv Chem Phys* **52**, 363.
8. Berg HC (1993) *Random Walks in Biology*, expanded edition. Princeton University Press.
9. McGrady GA, Marrow C, Myers G, Daniels M, Vera M, Mueller C, Liebow E and Klovdahl A, et al (1995) A note on implementation of a random-walk design to study adolescent social networks. *Social Networks* **17** (3-4), 251-255.
10. Havlin S and Ben-Avraham D (1987) Diffusion in disordered media. *Adv Phys* **36**, 695.
11. Newman TJ and Triampo W (1999) Binary data corruption due to a Brownian agent. *Phys Rev E* **59** (5), 5172.
12. Triampo W and Newman TJ (1999) Binary data corruption due to a Brownian agent. *Phys Rev E* **60** (2), 1450.
13 Gardiner CW (1985) *Handbook of Stochastic Processes*, 2nd ed. Springer-Verlag, Berlin.
14. Abramowitz M and Stegun IA (1972) *Handbook of Mathematical Functions*, 10th ed. Dover, NY.
15. Wolfram S (1987) *Theory and Applications of Cellular Automata*. World Scientific, Singapore.